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Quantum integrability in (super) Yang-Mills theory on the light-cone

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Abstract

We employ the light-cone formalism to construct in the (super) Yang-Mills theories in the multi-color limit the one-loop dilatation operator acting on single trace products of chiral superfields separated by light-like distances. In the $\mathcal{N} = 4$ Yang-Mills theory it exhausts all Wilson operators of the maximal Lorentz spin while in nonsupersymmetric Yang-Mills theory it is restricted to the sector of maximal helicity gluonic operators. We show that the dilatation operator in all \mathcal{N} -extended super Yang-Mills theories is given by the same integral operator which acts on the $(\mathcal{N} + 1)$ -dimensional superspace and is invariant under the $SL(2|\mathcal{N})$ superconformal transformations. We construct the R -matrix on this space and identify the dilatation operator as the Hamiltonian of the Heisenberg $SL(2|\mathcal{N})$ spin chain.

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1. Introduction

It has been recently recognized that four-dimensional Yang-Mills theory possesses a hidden symmetry. It is not manifest in the classical Lagrangian and rather reveals itself on the quantum level through integrability properties of evolution equations governing the energy dependence of scattering amplitudes in the Regge limit [1, 2] and renormalization group equations for composite Wilson operators [3, 4, 5, 6]. In both cases, the evolution equations can be brought to the form of a Schrödinger equation which contains a large enough number of hidden integrals of motion to make it completely integrable. The underlying integrable structures have been identified as the celebrated Heisenberg spin magnet and its generalization. Their stringy interpretation has been proposed in [7].

In this paper, we shall study renormalization of Wilson operators in (super) Yang-Mills theories. These operators mix under renormalization and the corresponding mixing matrix defines a representation of the dilatation operator [8]. The integrability of the latter has been discovered in QCD in the sector of gauge-invariant Wilson operators of the maximal helicity, the so-called quasipartonic operators [9]. To one-loop order, the dilatation operator inherits the conformal symmetry of the classical QCD Lagrangian and for the Wilson operators of the maximal Lorentz spin the full conformal group is reduced to the $SL(2)$ subgroup. In the multi-color limit, the mixing problem for such operators can be reformulated as an eigenvalue problem for a one-dimensional chain of $SL(2)$ conformal spins resided on sites of original QCD fields. Depending on the field content of Wilson operators, two integrable structures have been revealed: closed spin chains for multi-quark and multi-gluon operators, and open spin chains for operators involving fundamental matter at the ends of an array of gluons.

Similar integrability phenomena have been recently found in the context of maximally supersymmetric $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. There, the one-loop mixing matrix in the sector of scalar operators has been identified in the multi-color limit as a Hamiltonian of the Heisenberg $SO(6)$ spin chain [10, 11] and its generalization to arbitrary Wilson operators led to the identification of the $\mathcal{N} = 4$ dilatation operator with the Hamiltonian of the $PSU(2,2|4)$ super-spin chain [12]¹.

A natural question arises whether integrability phenomena found in QCD and in the $\mathcal{N} = 4$ SYM theory in the different sectors of Wilson operators are related to each other. In this paper, we shall argue that integrability is not a genuine new symmetry of the $\mathcal{N} = 4$ SYM theory but is a general feature of the Yang-Mills theory in the multi-color limit, at least to one-loop order. In particular, we shall demonstrate how the $\mathcal{N} = 4$ integrable structures are related to those found previously in QCD to which we shall refer as the $\mathcal{N} = 0$ SYM theory. We shall employ the light-cone formalism to construct in the (super) Yang-Mills theories in the multi-color limit the one-loop dilatation operator acting on single trace products of chiral superfields separated by light-like distances. In the $\mathcal{N} = 4$ theory it exhausts all Wilson operators of the maximal Lorentz spin while in $\mathcal{N} = 0$ theory it is restricted to the sector of maximal helicity gluonic operators. The advantage of this formalism is that the superfields involve only “physical”, propagating modes and superconformal transformations can be realized linearly on the product of superfields “living” on the light-cone. This allows one to construct the one-loop dilatation operator as a quantum mechanical Hamiltonian and map it into the Heisenberg $SL(2|\mathcal{N})$ spin chain.

The paper is organized as follows. In Sect. 2 we review the light-cone formulation of the

¹For super Yang-Mills theories with less supersymmetry analogous analyses have been performed in Refs. [13, 14, 15].

(super) Yang-Mills theory. In Sect. 3 we define the generating function for Wilson operators of the maximal Lorentz spin and formulate the evolution equation governing its scale dependence. The constraints imposed on the evolution equation by the superconformal symmetry are discussed in Sect. 4. The one-loop dilatation operator entering this equation is calculated in the \mathcal{N} -extended SYM in Sect. 5. It is identified in Sect. 6 as a Hamiltonian of the Heisenberg $SL(2|\mathcal{N})$ spin chain. Section 7 contains concluding remarks.

2. Yang-Mills theory on the light-cone

A convenient framework for discussing integrability properties of (super) Yang-Mills theories is provided by the “light-cone formalism” [16, 17, 18]. In this formalism, one sacrifices the full manifest covariance of the theory under the Poincaré transformations with advantage of having the possibility to integrate out non-propagating components of fields and formulate the quantum action in terms of “physical” degrees of freedom. Another benefit of using the light-cone formalism in the SYM theory is that the \mathcal{N} -extended supersymmetric algebra is closed off-shell for the propagating fields and there is no need to introduce auxiliary fields. This allows one to design an extended superspace formulation of $\mathcal{N} = 4$ SYM theory while the covariant form of the same theory does not exist.

In the light-cone formalism, one splits the gauge field into longitudinal and transverse components $A_\mu^a(x) = (A_+^a, A_-^a, A^a, \bar{A}^a)$ (with $A_\pm(x) \equiv \frac{1}{\sqrt{2}}(A_0(x) \pm A_3(x))$, $A(x) \equiv \frac{1}{\sqrt{2}}(A_1(x) + iA_2(x))$ and $\bar{A} = A^*$) and quantizes the SYM theory in a noncovariant, light-cone gauge $A_+^a(x) = 0$. Making a similar decomposition of (Majorana) fermion fields into the so-called “bad” and “good” components $\Psi = \Pi_+\Psi + \Pi_-\Psi \equiv \Psi_+ + \Psi_-$ (with $\Pi_\pm = \frac{1}{2}\gamma_\pm\gamma_\mp$), one finds that the fields $\Psi_-(x)$ and $A_-(x)$ can be integrated out in this gauge. The resulting action of SYM theory is expressed in terms of physical fields—transverse components of the gauge fields, $A(x)$ and $\bar{A}(x)$, “good” components of fermion fields $\Psi_+(x)$ and, in general, scalar fields. Let us summarize the light-cone formulation of SYM theories by starting with the $\mathcal{N} = 4$ model and going down to $\mathcal{N} = 2$, $\mathcal{N} = 1$ and, ultimately, $\mathcal{N} = 0$ models.

In the $\mathcal{N} = 4$ model, the propagating modes are the complex field $A(x)$ describing transverse components of the gauge field, three complex scalar fields $\phi^{AB}(x)$, complex Grassmann fields $\lambda^A(x)$ defining “good” components of four Majorana fermions (with $A, B = 1, \dots, 4$), and conjugated to them are fields $\bar{A}(x)$, $\bar{\phi}_{AB} = (\phi^{AB})^* = \frac{1}{2}\varepsilon_{ABCD}\phi^{CD}$ and $\bar{\lambda}_A(x)$. It is tacitly assumed that the fields belong to the adjoint representation of the $SU(N_c)$ group. In the light-cone formalism, all propagating fields can be combined into a single scalar superfield $\Phi(x_\mu, \theta^A, \bar{\theta}_A)$, and the action of the $\mathcal{N} = 4$ SYM reads as [17]

$$S_{\mathcal{N}=4} = \int d^4x d^4\theta d^4\bar{\theta} \left\{ \frac{1}{2} \bar{\Phi}^a \square \Phi^a - \frac{2}{3} g f^{abc} \left(\frac{1}{\partial_+} \bar{\Phi}^a \Phi^b \bar{\partial} \Phi^c + \frac{1}{\partial_+} \Phi^a \bar{\Phi}^b \partial \bar{\Phi}^c \right) - \frac{1}{2} g^2 f^{abc} f^{ade} \left(\frac{1}{\partial_+} (\Phi^b \partial_+ \bar{\Phi}^c) \frac{1}{\partial_+} (\bar{\Phi}^d \partial_+ \bar{\Phi}^e) + \frac{1}{2} \Phi^b \bar{\Phi}^c \Phi^d \bar{\Phi}^e \right) \right\}, \quad (2.1)$$

where f^{abc} are the $SU(N_c)$ structure constants, $\partial_+ = \frac{1}{\sqrt{2}}(\partial_{x_0} - \partial_{x_3})$, $\partial = \frac{1}{\sqrt{2}}(\partial_{x_1} + i\partial_{x_2})$, $\bar{\partial} = (\partial)^*$ and the integration measure over Grassmann variables is normalized as $\int d^N\theta \theta^1 \dots \theta^N =$

$\int d^{\mathcal{N}} \bar{\theta} \bar{\theta}_1 \dots \bar{\theta}_{\mathcal{N}} = 1$. The complex scalar $\mathcal{N} = 4$ superfield is defined as

$$\Phi(x, \theta^A, \bar{\theta}_A) = e^{\frac{1}{2} \bar{\theta} \cdot \theta \partial_+} \left\{ \partial_+^{-1} A(x) + \theta^A \partial_+^{-1} \bar{\lambda}_A(x) + \frac{i}{2!} \theta^A \theta^B \bar{\phi}_{AB}(x) \right. \\ \left. - \frac{1}{3!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \lambda^D(x) - \frac{1}{4!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \theta^D \partial_+ \bar{A}(x) \right\}. \quad (2.2)$$

Here $\bar{\theta} \cdot \theta = \bar{\theta}_A \theta^A$ and nonlocal operator ∂_+^{-1} is defined using the Mandelstam-Leibbrandt prescription [18]. We recall that the superfield (2.2) comprises *all* propagating fields, and expansion in θ^A can be viewed as an expansion in different helicity components: +1 for $A(x)$, 1/2 for $\bar{\lambda}_A(x)$, 0 for $\bar{\phi}_{AB}$, -1/2 for $\lambda^A(x)$ and -1 for $\bar{A}(x)$. Notice that the two lowest components in (2.2) are nonlocal operators. A unique feature of (2.2) is that conjugated superfield is not independent and is related to $\Phi(x, \theta^A, \bar{\theta}_A)$ as

$$\bar{\Phi}(x, \theta^A, \bar{\theta}_A) = -\partial_+^{-2} D_1 D_2 D_3 D_4 \Phi(x, \theta^A, \bar{\theta}_A) = -\frac{1}{4!} \partial_+^{-2} \varepsilon^{ABCD} D_A D_B D_C D_D \Phi(x, \theta^A, \bar{\theta}_A). \quad (2.3)$$

Here the notation was introduced for the covariant derivatives in the superspace

$$D_A = \partial_{\theta^A} - \frac{1}{2} \bar{\theta}_A \partial_+, \quad \bar{D}^A = \partial_{\bar{\theta}_A} - \frac{1}{2} \theta^A \partial_+, \quad \{D_A, \bar{D}^B\} = -\delta_A^B \partial_+. \quad (2.4)$$

The superfields (2.2) and (2.3) satisfy the chirality condition

$$\bar{D}^A \Phi(x, \theta^A, \bar{\theta}_A) = D_A \bar{\Phi}(x, \theta^A, \bar{\theta}_A) = 0. \quad (2.5)$$

As was first found in Refs. [19, 17, 20, 18], all Green's functions computed from (2.1) do not contain ultraviolet divergences to all orders of perturbation theory and, therefore, the $\mathcal{N} = 4$ light-cone action (2.1) defines an ultraviolet finite quantum field theory. As such, it inherits all symmetries of the classical Lagrangian including the invariance under superconformal transformations. Later in Sect. 4 we shall make use of the subgroup of these transformations that leaves the '+'-direction on the light-cone invariant.

Let us now turn to Yang-Mills theories with less supersymmetry. Their light-cone formulation can be obtained from $\mathcal{N} = 4$ SYM using the “method of truncation” [21] which is based on the following identity

$$\int d^4 x d^{\mathcal{N}} \theta d^{\mathcal{N}} \bar{\theta} \mathcal{L}(\Phi) = (-1)^{\mathcal{N}} \int d^4 x d^{\mathcal{N}-1} \theta d^{\mathcal{N}-1} \bar{\theta} [\bar{D}^{\mathcal{N}} D_{\mathcal{N}} \mathcal{L}(\Phi)] \Big|_{\theta^{\mathcal{N}} = \bar{\theta}_{\mathcal{N}} = 0}. \quad (2.6)$$

Applying (2.6), one can rewrite the $\mathcal{N} = 4$ model in terms of one $\mathcal{N} = 2$ light-cone Yang-Mills chiral superfield $\Phi^{(2)}(x, \theta^A, \bar{\theta}_A)$ coupled to the $\mathcal{N} = 2$ Wess-Zumino chiral superfield $\Psi^{(2)}(x, \theta^A, \bar{\theta}_A)$

$$\Phi^{(2)} = \Phi^{(4)}(x, \theta^A, \bar{\theta}_A) \Big|_{\substack{\theta^3 = \bar{\theta}_3 = 0 \\ \theta^4 = \bar{\theta}_4 = 0}}, \quad \Psi^{(2)} = D_3 \Phi^{(4)}(x, \theta^A, \bar{\theta}_A) \Big|_{\substack{\theta^3 = \bar{\theta}_3 = 0 \\ \theta^4 = \bar{\theta}_4 = 0}}, \quad (2.7)$$

where the superscript refers to the underlying \mathcal{N} -extended SYM and $\Phi^{(4)}$ is defined in (2.2). Putting $\Psi^{(2)} = 0$, one obtains the light-cone formulation of the $\mathcal{N} = 2$ SYM [21, 22]

$$S_{\mathcal{N}=2} = \int d^4 x d^2 \theta d^2 \bar{\theta} \left\{ -\bar{\Phi}^a \square \Phi^a + 2g f^{abc} (\partial_+ \Phi^a \bar{\Phi}^b \bar{\partial} \Phi^c + \partial_+ \bar{\Phi}^a \Phi^b \partial \bar{\Phi}^c) \right. \\ \left. - 2g^2 f^{abc} f^{ade} \frac{1}{\partial_+} (\partial_+ \Phi^b \bar{D}^1 \bar{D}^2 \bar{\Phi}^c) \frac{1}{\partial_+} (\partial_+ \bar{\Phi}^d D_1 D_2 \Phi^e) \right\}. \quad (2.8)$$

Here $\Phi \equiv \Phi^{(2)}(x, \theta^A, \bar{\theta}_A)$ is a complex chiral $\mathcal{N} = 2$ superfield. Substituting (2.2) into (2.7) one finds

$$\Phi(x, \theta^A, \bar{\theta}_A) = e^{\frac{1}{2}\bar{\theta}\theta\partial_+} \left\{ \partial_+^{-1} A(x) + \theta^A \partial_+^{-1} \bar{\lambda}_A(x) + \frac{i}{2!} \varepsilon_{AB} \theta^A \theta^B \bar{\phi}(x) \right\}, \quad (2.9)$$

where $\bar{\phi} \equiv \bar{\phi}_{12}(x)$ and $A, B = 1, 2$. The conjugated (anti-chiral) superfield $\bar{\Phi}(x, \theta^A, \bar{\theta}_A)$ involves the fields $\bar{A}(x)$, λ^A and ϕ , and in distinction with the $\mathcal{N} = 4$ model, it is independent on $\Phi^{(2)}(x, \theta^A, \bar{\theta}_A)$. In the $\mathcal{N} = 2$ light-cone action (2.8), the propagating fields are the transverse components of the gauge field $A(x)$, one complex scalar field $\phi(x)$ and two complex Grassmann fields $\lambda^A(x)$ describing “good” components of two Majorana fermions.

As a next step, one applies (2.6) to truncate the $\mathcal{N} = 2$ down to $\mathcal{N} = 1$ SYM. Similar to the previous case, one defines two chiral superfields $\Phi^{(1)} = \Phi^{(2)}(x, \theta^A, \bar{\theta}_A)|_{\theta^2=\bar{\theta}_2=0}$ and $\Psi^{(1)} = D_2 \Phi^{(2)}(x, \theta^A, \bar{\theta}_A)|_{\theta^2=\bar{\theta}_2=0}$ and puts $\Psi^{(1)} = 0$ to retain only the contribution of the $\mathcal{N} = 1$ SYM superfield. This leads to

$$S_{\mathcal{N}=1} = \int d^4x d\theta d\bar{\theta} \left\{ \bar{\Phi}^a \square \partial_+ \Phi^a + 2g f^{abc} (\partial_+ \Phi^a \partial_+ \bar{\Phi}^b \bar{\partial} \Phi^c - \partial_+ \bar{\Phi}^a \partial_+ \Phi^b \partial \bar{\Phi}^c) \right. \\ \left. + 2g^2 f^{abc} f^{ade} \frac{1}{\partial_+} (\partial_+ \Phi^b \bar{D}^1 \partial_+ \bar{\Phi}^c) \frac{1}{\partial_+} (\partial_+ \bar{\Phi}^d D_1 \partial_+ \Phi^e) \right\}, \quad (2.10)$$

where the $\mathcal{N} = 1$ light-cone chiral superfield $\Phi \equiv \Phi^{(1)}(x, \theta, \bar{\theta})$ is given by

$$\Phi(x, \theta, \bar{\theta}) = e^{\frac{1}{2}\bar{\theta}\theta\partial_+} \left\{ \partial_+^{-1} A(x) + \theta \partial_+^{-1} \bar{\lambda}(x) \right\}, \quad (2.11)$$

with $\bar{\lambda} = \bar{\lambda}_1(x)$. In the $\mathcal{N} = 1$ light-cone action (2.10), the propagating fields are the transverse components of the gauge fields $A(x)$ and one complex Grassmann field $\lambda(x)$ describing “good” component of Majorana fermion.

Finally, we use (2.6) to truncate $\mathcal{N} = 1$ down to $\mathcal{N} = 0$ Yang-Mills theory. The resulting light-cone action takes the form

$$S_{\mathcal{N}=0} = \int d^4x \left\{ \bar{\Phi}^a \square \partial_+^2 \Phi^a - 2g f^{abc} (\partial_+ \Phi^a \partial_+^2 \bar{\Phi}^b \bar{\partial} \Phi^c + \partial_+ \bar{\Phi}^a \partial_+^2 \Phi^b \partial \bar{\Phi}^c) \right. \\ \left. - 2g^2 f^{abc} f^{ade} \frac{1}{\partial_+} (\partial_+ \Phi^b \partial_+^2 \bar{\Phi}^c) \frac{1}{\partial_+} (\partial_+ \bar{\Phi}^d \partial_+^2 \Phi^e) \right\}, \quad (2.12)$$

where the $\mathcal{N} = 0$ field is given by

$$\Phi(x) = \Phi^{(1)}(x, \theta, \bar{\theta})|_{\theta=\bar{\theta}=0} = \partial_+^{-1} A(x). \quad (2.13)$$

The light-cone action (2.12) coincides with the well-known expression for the action of $SU(N_c)$ gluodynamics quantized in the gauge $A_+(x) = 0$ [23].

It seems like an overcomplication to work with (2.13) since one can easily reformulate the action (2.12) in terms of a local, field strength tensor $\partial_+ A(x)$. In a similar manner, the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ light-cone actions, Eqs. (2.10) and (2.8), can be formulated in terms of superfields, $\Phi_{\text{new}}^{(1)} = e^{\frac{1}{2}\bar{\theta}\theta\partial_+} \{\bar{\lambda}(x) + \theta \partial_+ A(x)\}$ and $\Phi_{\text{new}}^{(2)} = e^{\frac{1}{2}\bar{\theta}\theta\partial_+} \{i\bar{\phi}(x) + \theta^A \bar{\lambda}_A(x) - \frac{1}{2!} \varepsilon_{AB} \theta^A \theta^B \partial_+ A(x)\}$, respectively, expressed solely in terms of the elementary conformal primaries. In the $\mathcal{N} = 4$ case, this is not possible on the basis of simple dimensional considerations alone since the superfield has to embrace all available particle helicity states. However, in our analysis we prefer to deal with the superfields involving nonlocal operators. The reason for this is that as we will show below the evolution operator governing the scale dependence of the product of such superfields turns out to be universal for all Yang-Mills theories in question.

3. Evolution equations on the light-cone

Let us consider the scale dependence of Wilson operators in the \mathcal{N} -extended SYM. In general, these are local gauge-invariant composite operators $\mathbb{O}_{\mu_1\mu_2\dots\mu_L}(x)$ built from fundamental fields – strength tensor $F_{\mu\nu}(x)$, fermions $\Psi^A(x)$, scalars $\phi^{AB}(x)$ – and their complex conjugate as well as arbitrary number of covariant derivatives $D_\mu = \partial_\mu - ig[A_\mu(x), \]$ acting on them. The Wilson operators carry Lorentz and isotopic $SU(\mathcal{N})$ indices and can be classified according to irreducible representations of both groups. Later in this section we shall study the Wilson operators of the maximal Lorentz spin or, equivalently, the minimal twist. They have the Lorentz structure completely symmetric and traceless in any pair of Lorentz indices. Such operators can be obtained by projecting a general operator onto light-like vectors $\mathbb{O}_L^{(\max)} = n^{\mu_1} n^{\mu_2} \dots n^{\mu_L} \mathbb{O}_{\mu_1\mu_2\dots\mu_L}(x)$ with $n_\mu^2 = 0$.

In the light-cone formalism, the Wilson operators can be built only from “physical” components of fundamental fields (transverse components of gauge fields, “good” components of fermions and scalars) and covariant derivatives acting on them. As was already mentioned, the remaining components are not dynamically independent and can be expressed (nonlocally) in terms of physical ones by virtue of the equations of motion. In addition, making use of the superfield formulation, one can construct the Wilson operators directly from the light-cone scalar chiral $\Phi(x, \theta^A, \bar{\theta}_A)$ and antichiral $\bar{\Phi}(x, \theta^A, \bar{\theta}_A)$ superfields, covariant derivatives D_μ and derivatives acting on Grassmann variables ∂_{θ^A} and $\partial_{\bar{\theta}_A}$. This construction holds in the SYM theory regardless the number of supersymmetries involved. Notice that according to (2.3) the Wilson operators in the $\mathcal{N} = 4$ SYM can be built solely from chiral superfields. For $\mathcal{N} \leq 2$ the superfields $\Phi(x, \theta^A, \bar{\theta}_A)$ and $\bar{\Phi}(x, \theta^A, \bar{\theta}_A)$ are independent on each other and have to be taken into account on equal footing.

In what follows we shall restrict ourselves to the subclass of single trace maximal spin Wilson operators built entirely from chiral (or anti-chiral) superfields. In the light-like axial gauge $n \cdot A \equiv A_+(x) = 0$, such operators can be constructed from the strength tensor $n_\mu F^\mu{}_\nu = F_{+\nu}(x) = \partial_+ A_\nu(x)$, or equivalently, $\partial_+ A(x)$ and $\partial_+ \bar{A}(x)$, fermions (λ^A and $\bar{\lambda}_A$), scalars (ϕ^{AB} and $\bar{\phi}_{AB}$) and covariant derivatives $n \cdot D = D_+ = \partial_+$. The generating function for such operators takes the form

$$\mathbb{O}(Z_1, Z_2, \dots, Z_L) = \text{tr} \left\{ \Phi(x_1 n_\mu, \theta_1^A, \bar{\theta}_{1A}) \Phi(x_2 n_\mu, \theta_2^A, \bar{\theta}_{2A}) \dots \Phi(x_L n_\mu, \theta_L^A, \bar{\theta}_{LA}) \right\}. \quad (3.1)$$

Here all superfields are located on the light-cone along the ‘+’-direction defined by a light-like vector n_μ ($n_+ = n_\perp = 0$ and $n_- = 1$) and the variables x_k specify their position. The Z_k -variables in the left-hand side of (3.1) stand for the coordinates of the superfields in the superspace. According to (2.2), a chiral superfield satisfies the relation $\Phi(x n_\mu, \theta^A, \bar{\theta}_A) = \Phi(n_\mu(x + \frac{1}{2}\bar{\theta} \cdot \theta), \theta^A, 0)$ which allows one to eliminate the $\bar{\theta}$ -dependence in (3.1) and define the Z -coordinates in the superspace as ²

$$Z = (z, \theta^1, \dots, \theta^\mathcal{N}), \quad z \equiv x + \frac{1}{2}\bar{\theta} \cdot \theta. \quad (3.2)$$

Later we shall use a shorthand notation for the chiral light-cone superfield $\Phi(Z) = \Phi(z, \theta^A)$.

The expansion of the right-hand side of (3.1) in powers of $z_j - z_k$ yields an infinite tower of operators

$$\prod_{l=1}^L (\partial_{z_l})^{k_l} \mathbb{O}(Z_1, \dots, Z_L) \Big|_{z_l=0} = \text{tr} \left\{ D_+^{k_1} \Phi(0, \theta_1^A) \dots D_+^{k_L} \Phi(0, \theta_L^A) \right\}, \quad (3.3)$$

²Throughout the paper we adopt the following convention for the complex conjugation $(\bar{\theta}\chi)^* = \theta\bar{\chi}$.

where we took into account that $n \cdot D = D_+ = \partial_+$ in the gauge $A_+(x) = 0$. Expanding further the right-hand side of (3.3) in powers of θ^A -variables, one obtains a set of composite operators built from propagating fields. Notice that not all of them are Wilson operators. The reason for this is that the superfields, Eqs. (2.2), (2.9), (2.11) and (2.13), involve nonlocal operators $\partial_+^{-1} A(x)$ and $\partial_+^{-1} \bar{\lambda}_A(x)$, while the Wilson operators can only involve $\partial_+ A(x)$, $\bar{\lambda}_A(x)$ and their derivatives. For instance, expansion of the light-cone operator $\mathbb{O}(Z_1, Z_2)$ in the $\mathcal{N} = 4$ theory gives rise to the following dimension two operators: $\phi^{AB} \bar{\phi}_{AB}(0)$, $\partial_+^{-1} A \partial_+ \bar{A}(0)$ and $\partial_+^{-1} \bar{\lambda}_A \lambda_A(0)$. Among them only the first one is a Wilson operator. Although naively one might expect that this operator could mix under renormalization with the other two, the locality of the $\mathcal{N} = 4$ theory prohibits such mixing.

To eliminate the contribution of nonlocal operators to (3.1), one has to project the nonlocal operator (3.1) onto a “physical” subspace of *local* Wilson operators

$$\mathbb{O}^w(Z_1, \dots, Z_L) = \Pi \cdot \mathbb{O}(Z_1, \dots, Z_L) \equiv \text{tr} \{ \Phi^w(z_1, \theta_1^A) \dots \Phi^w(z_L, \theta_L^A) \} , \quad (3.4)$$

where the notation was introduced for the superfield

$$\Phi^w(z, \theta^A) = \Phi(z, \theta^A) - \Phi(0, 0) - Z \cdot \partial_Z \Phi(0, 0) , \quad (3.5)$$

with $Z \cdot \partial_Z \equiv z \partial_z + \theta^A \partial_{\theta^A}$. The superfield $\Phi^w(z, \theta^A)$ does not involve nonlocal gauge and fermion operators, $\partial_+^{-1} A(0)$, $A(0)$ and $\partial_+^{-1} \bar{\lambda}^A(0)$, and, therefore, $\mathbb{O}^w(Z_1, \dots, Z_L)$ generates only Wilson operators. It is easy to verify using (3.4) that Π is a projector, $\Pi^2 = \Pi$.

The Wilson operators generated by the light-cone operator (3.4) mix under renormalization. To find their anomalous dimensions one has to diagonalize the corresponding mixing matrix. For the light-cone SYM theories introduced in the previous section, this matrix can be calculated in perturbation theory using the supergraph Feynman technique developed in Ref. [17]. The size of the mixing matrix depends on the total number of derivatives in (3.3) and it rapidly grows as this number increases. To avoid such complication and in order to reveal a hidden symmetry of the evolution equations, it is convenient to work directly with nonlocal light-cone operators (3.1). Having established the scale dependence of the light-cone operator (3.1), one can reconstruct the mixing matrix by substituting the nonlocal operator $\mathbb{O}(Z_1, Z_2, \dots, Z_L)$ by its expansion in terms of local Wilson operators (3.3).

In the multi-color limit, $N_c \rightarrow \infty$ and $g^2 N_c = \text{fixed}$, to the lowest order of perturbation theory, quantum corrections to (3.1) arise due to interaction between two nearest neighbor superfields $\Phi(Z_k)$ and $\Phi(Z_{k+1})$ (with $k = 1, \dots, L$). The corresponding Feynman diagrams are displayed in Fig. 1. They involve cubic and quartic vertices which can be read off the light-cone actions, Eqs. (2.1), (2.8), (2.10) and (2.12). The first two diagrams are divergent due to light-cone separation of the superfields while the remaining two diagrams contain conventional ultraviolet divergences due to the renormalization of the superfields. To one-loop order the renormalization group (Callan-Symanzik) equation for the nonlocal operator (3.1) can be written as [9, 24]

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right\} \mathbb{O}(Z_1, Z_2, \dots, Z_L) = -\frac{g^2 N_c}{8\pi^2} [\mathbb{H} \cdot \mathbb{O}](Z_1, Z_2, \dots, Z_L) + \mathcal{O}((g^2 N_c)^2) , \quad (3.6)$$

where \mathbb{H} is some integral operator and $\beta(g) = 0$ in the $\mathcal{N} = 4$ SYM theory. This equation is just a Ward identity for the dilatation operator in the \mathcal{N} -extended SYM theory in the multi-color limit (see, e.g., [25]). The evolution kernel \mathbb{H} defines the representation of the dilatation operator

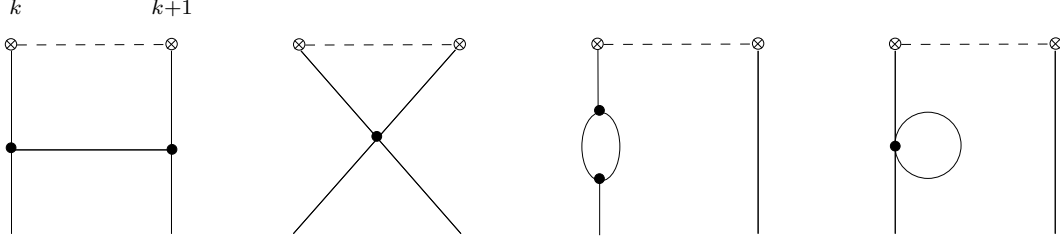


Figure 1: Feynman supergraphs defining one-loop contribution to the two-particle evolution kernel $\mathbb{H}_{k,k+1}$ in the multi-color limit. The dashed line denotes the ‘+’-direction on the light-cone and the open circles define the position of the superfields.

on the space of light-cone superfields (3.1). Its eigenvalues determine the spectrum of anomalous dimensions of Wilson operators in the \mathcal{N} -extended SYM theory.

The possible form of the evolution kernel \mathbb{H} is constrained by the symmetries of the underlying SYM theory. To one-loop order, in the multi-color limit the operator \mathbb{H} can be written as the sum over two-particle nearest neighbor interactions

$$\mathbb{H} = \mathbb{H}_{12} + \dots + \mathbb{H}_{L-1,L} + \mathbb{H}_{L,1}. \quad (3.7)$$

The two-particle kernel $\mathbb{H}_{k,k+1}$ receives contribution from the Feynman diagrams shown in Fig. 1. We will establish the explicit form of \mathbb{H} in the \mathcal{N} -extended SYM theory in Sect. 5.

Secondly, we recall that \mathbb{H} acts on the product of superfields (3.1) whose expansion around $Z_k = 0$ generates both Wilson operators and nonlocal operators. The OPE implies that the Wilson operators cannot mix with nonlocal operators while the inverse is possible. Thus, the evolution kernel \mathbb{H} in (3.6) has to transform “physical” operators $\mathbb{O}^w(Z_1, \dots, Z_L)$, Eq. (3.4), into themselves. To satisfy this requirement it suffices to demand that $\Pi \mathbb{H} (1 - \Pi) = 0$ where the projector Π was introduced in Eq. (3.4). We shall check this relation in Sect. 5 (see Eq. (5.4) below).

Finally, the third constraint is imposed by the invariance of the classical action of the light-cone SYM theory, Eqs. (2.1), (2.8), (2.10) and (2.12), under superconformal transformations. This symmetry survives at the quantum level in the $\mathcal{N} = 4$ theory, while at $\mathcal{N} = 2$, $\mathcal{N} = 1$ and $\mathcal{N} = 0$ it is broken by quantum corrections. In the latter case, however, symmetry breaking effects affect the evolution equations (3.6) starting only from the two-loop level [26]. This means that the one-loop operator \mathbb{H} possesses the symmetry of the classical action and, therefore, it has to commute with the generators of the \mathcal{N} -extended superconformal group. As we will show in the next section, this imposes severe restrictions on the possible form of the evolution kernel (3.7).

4. Superconformal symmetry on the light-cone

The \mathcal{N} -extended superconformal algebra $SU(2, 2|\mathcal{N})$ contains 15 even charges \mathbf{P}_μ , $\mathbf{M}_{\mu\nu}$, \mathbf{D} and \mathbf{K}_μ and $4\mathcal{N}$ odd charges $\mathbf{Q}_{\alpha A}$, $\bar{\mathbf{Q}}^{\dot{\alpha} A}$, \mathbf{S}_α^A , $\bar{\mathbf{S}}_A^{\dot{\alpha}}$ which are two-component Weyl spinors and $A = 1, 2, \dots, \mathcal{N}$. It also contains additional bosonic chiral charge \mathbf{R} and, in case of extended $\mathcal{N} > 1$ supersymmetries, the charges \mathbf{T}_A^B satisfying the $SU(\mathcal{N})$ commutation relations [27]. Scalar light-cone superfield $\Phi(x, \theta^A, \bar{\theta}_A)$ realizes a representation of this algebra. Its infinitesimal

variations under superconformal transformations look like

$$\delta^G \Phi(x, \theta^A, \bar{\theta}_A) = i[\Phi(x, \theta^A, \bar{\theta}_A), \mathbf{G}] = -iG \Phi(x, \theta^A, \bar{\theta}_A), \quad (4.1)$$

where $\mathbf{G} = \varepsilon^\mu \mathbf{P}_\mu, \varepsilon^{\mu\nu} \mathbf{M}_{\mu\nu}, \dots$ for even generators and $\mathbf{G} = \xi \mathbf{Q}, \dots$ for odd generators with ξ being a constant Grassmann-valued Weyl spinor. In this representation, the quantum-field operator \mathbf{G} is represented by the operator G acting on even and odd coordinates of the superfield [27].

Let us examine the action of superconformal algebra on the operators (3.1). We remind that the superfields entering (3.1) are located on the light-cone along the ‘+’-direction defined by the light-like vector n_μ . In order to preserve this property, we have to restrict ourselves to superconformal transformations that map the ‘+’-direction on the light-cone into itself. Under these restrictions the full superconformal algebra is reduced to its subalgebra, the so-called collinear $SL(2|\mathcal{N})$ superalgebra, containing the following generators

$$\mathbf{P}_+, \quad \mathbf{M}_{-+}, \quad \mathbf{D}, \quad \mathbf{K}_-, \quad \mathbf{Q}_{+\alpha A}, \quad \bar{\mathbf{Q}}_+^{\dot{\alpha} A}, \quad \mathbf{S}_{-\alpha}^A, \quad \bar{\mathbf{S}}_{-A}^{\dot{\alpha}}, \quad \mathbf{R}, \quad \mathbf{T}_A^B, \quad \mathbf{M}_{12}, \quad (4.2)$$

where for the odd charges the $+/-$ subscript indicates “good”/“bad” components of the Weyl spinor.³ In the light-cone formalism, such one-component spinors can be described by a complex Grassmann field without any Lorentz index.

In the light-cone formalism, the action of the generators of the collinear superalgebra (4.2) on the chiral scalar superfield $\Phi(xn_\mu, \theta^A, \bar{\theta}_A)$ can be represented as differential operators acting on $z = x + \frac{1}{2} \bar{\theta} \cdot \theta$ and θ^A while the remaining generators of the superconformal algebra have a nonlinear realization on the superfield [21]. Introducing linear combinations of the operators $iP_+ \equiv -L^-$, $\frac{1}{2}iK_- \equiv L^+$, $\frac{1}{2}i(D + M_{-+}) \equiv L^0$, $i(D - M_{-+}) \equiv E$, $iQ_{+A} \equiv \sqrt[4]{8} V_A^-$, $i\bar{Q}_+^A \equiv -i\sqrt[4]{8} W^{A,-}$, $iS_-^A \equiv -\sqrt[4]{32} W^{A,+}$, $i\bar{S}_{-A} \equiv i\sqrt[4]{32} V_A^+$ and $\frac{1}{4}(1 - \frac{4}{\mathcal{N}})R + \frac{1}{2}M_{12} \equiv B$, we find that they admit the following representation for the chiral scalar superfield

$$\begin{aligned} L^- &= -\partial_z, & L^+ &= 2jz + z^2 \partial_z + z(\theta \cdot \partial_\theta), & L^0 &= j + z\partial_z + \frac{1}{2}(\theta \cdot \partial_\theta), & E &= t, \\ W^{A,-} &= \theta^A \partial_z, & W^{A,+} &= \theta^A [2j + z\partial_z + (\theta \cdot \partial_\theta)], & V_A^- &= \partial_{\theta^A}, & V_A^+ &= z\partial_{\theta^A}, \\ T_B^A &= \theta^A \partial_{\theta^B} - \frac{1}{\mathcal{N}} \delta_B^A (\theta \cdot \partial_\theta), & B &= -j - \frac{1}{2} \left(1 - \frac{2}{\mathcal{N}}\right) (\theta \cdot \partial_\theta). \end{aligned} \quad (4.3)$$

where $\partial_z \equiv \partial/\partial z$ and $\theta \cdot \partial_\theta \equiv \theta^A \partial/\partial \theta^A$. The operators (4.3) satisfy the $SL(2|\mathcal{N})$ (anti-) commutation relations [28]. In Eq. (4.3), $j = \frac{1}{2}(s + \ell)$ is the *conformal spin* and $t = \ell - s$ is the *twist* of the superfield [25]. Also, ℓ and s are correspondingly the canonical dimension and projection of the spin on the ‘+’-direction of the superfield defined as $i[\mathbf{D}, \Phi(0, 0, 0)] = \ell \Phi(0, 0, 0)$ and $i[\mathbf{M}_{-+}, \Phi(0, 0, 0)] = s \Phi(0, 0, 0)$. For the scalar superfields defined in (2.2), (2.9), (2.11) and (2.13) one has $\ell = 0$ and $s = -1$ leading to

$$j = -\frac{1}{2}, \quad t = 1. \quad (4.4)$$

The one-loop evolution kernel \mathbb{H} has to respect the superconformal symmetry and, therefore, it has to commute with the generators of the $SL(2|\mathcal{N})$ collinear superalgebra. Let us determine a

³For arbitrary Weyl spinors χ_α and $\bar{\chi}^{\dot{\alpha}}$ the projection onto the “good” and “bad” components looks like $\chi_{\pm\alpha} = \frac{1}{2} \bar{\sigma}^\mp_{\alpha\dot{\beta}} \sigma^\pm{}^{\dot{\beta}\gamma} \chi_\gamma$ and $\bar{\chi}_\pm^{\dot{\alpha}} \equiv \frac{1}{2} \sigma^\mp{}^{\alpha\beta} \bar{\sigma}^\pm{}_{\beta\dot{\gamma}} \bar{\chi}^{\dot{\gamma}}$.

general form of the operator satisfying these conditions. To begin with, we consider the evolution of the light-cone operator (3.1) in the special limit when all odd variables are set equal to zero,

$$\mathbb{O}^{(0)}(x_1, \dots, x_L) \equiv \mathbb{O}(Z_1, \dots, Z_L) \Big|_{\theta_1^A = \dots = \theta_L^A = 0} = \text{tr} \{ \Phi(x_1 n_\mu, 0) \dots \Phi(x_L n_\mu, 0) \}. \quad (4.5)$$

Since $\Phi(x n_\mu, 0) = \partial_x^{-1} A(x n_\mu)$, the operator (4.5) is reduced to the product of L gauge fields of helicity +1. To one-loop order, the operator (4.5) can only mix with the operators containing the same number of superfields and the same total helicity. This means that $\mathbb{O}^{(0)}(x_1, \dots, x_L)$ evolves autonomously under renormalization group transformations and, therefore, the evolution kernel \mathbb{H} has to map light-cone operators (4.5) into themselves. Notice that for the operators (4.5) the collinear superconformal group (4.3) is reduced to its $SL(2)$ subgroup with the generators $l^- = -\sum_k \partial_{x_k}$, $l^+ = \sum_k 2j x_k + x_k^2 \partial_{x_k}$ and $l^0 = \sum_k j + x_k \partial_{x_k}$. Therefore, the evolution kernel governing the scale dependence of the operators (4.5) has to be $SL(2)$ invariant.

A general form of the $SL(2)$ invariant kernel has been found in Ref. [4]. To one-loop order, in the multi-color limit it has the form (3.7) with the two-particle kernel given by

$$[\mathbb{H}_{12} \cdot \mathbb{O}^{(0)}](x_1, \dots, x_L) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta (\alpha\beta)^{2j-2} \varphi\left(\frac{\bar{\alpha}\bar{\beta}}{\alpha\beta}\right) \mathbb{O}^{(0)}(x_1 - \bar{\alpha}x_{12}, x_2 + \bar{\beta}x_{12}, x_3, \dots, x_L), \quad (4.6)$$

where $\bar{\alpha} = 1 - \alpha$, $\bar{\beta} = 1 - \beta$ and $x_{12} = x_1 - x_2$. Here j is the conformal spin of the fields entering (4.5) and φ is an arbitrary function.⁴ The operator (4.6) has a simple interpretation—acting on the product of L fields situated on the light-cone it displays only two of them (labeled as 1 and 2) in the direction of each other. The explicit form of the function $\varphi(\xi)$ is not fixed by the $SL(2)$ invariance. It depends both on the underlying gauge theory and the operator under consideration [4].

The two-particle kernel \mathbb{H}_{12} has to be invariant under the superconformal transformations and act locally on the superfields $\Phi(Z_1)$ and $\Phi(Z_2)$. This implies in particular that $[\mathbb{H}_{12}, V_A^{\pm, (1)} + V_A^{\pm, (2)}] = 0$ and $[\mathbb{H}_{12}, V_A^{\pm, (k)}] = 0$ where $k \geq 3$ and the superscript (k) indicates that the charges V_A^\pm defined in (4.3) act on the coordinates of k th field. These charges generate shifts in the superspaces along the odd directions

$$\mathbb{O}^{(0)}(x_1, x_2) = e^{-\epsilon^A [V_A^{-, (1)} + V_A^{-, (2)}] - \chi^A [V_A^{+, (1)} + V_A^{+, (2)}]} \mathbb{O}(Z_1, Z_2), \quad (4.7)$$

where $\chi^A = (\theta_1^A - \theta_2^A)/x_{12}$ and $\epsilon^A = (x_1 \theta_2^A - x_2 \theta_1^A)/x_{12}$. Combining together (4.7) and (4.6) one gets

$$[\mathbb{H}_{12} \cdot \mathbb{O}](Z_1, \dots, Z_L) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta (\alpha\beta)^{2j-2} \varphi\left(\frac{\bar{\alpha}\bar{\beta}}{\alpha\beta}\right) \mathbb{O}(Z_1 - \bar{\alpha}Z_{12}, Z_2 + \bar{\beta}Z_{12}, Z_3, \dots, Z_L), \quad (4.8)$$

where $Z_{12} = Z_1 - Z_2 \equiv (x_1 - x_2, \theta_1^1 - \theta_2^1, \dots, \theta_1^N - \theta_2^N)$. As before, the superconformal symmetry does not allow one to fix the explicit form of the weight function $\varphi(\xi)$.

To summarize, Eq. (4.8) defines the most general form of the two-particle evolution kernel consistent with the symmetries of \mathcal{N} –extended SYM theory. This kernel has a transparent interpretation in the superspace. Acting on the light-cone operator (3.1) it displaces the superfields

⁴Strictly speaking, this formula holds only for $j \geq 1/2$ while for $j < 1/2$ it has to be modified to make the integral convergent (see below).

located at the points Z_1 and Z_2 in the direction of each other. As we will show in the next section, the fact that both even and odd coordinates of the superfields are modified simultaneously implies certain pattern of the mixing between Wilson operators.

5. One-loop dilatation operator

The expression for the two-particle evolution kernel (4.8) depends on the conformal spin of the superfield $j = -1/2$, Eq. (4.4), and yet unknown function $\varphi(\xi)$. To determine this function one has to calculate the Feynman supergraphs shown in Fig. 1 and match their divergent part into a general expression for the two-particle kernel \mathbb{H}_{12} , Eq. (4.8).

Going through the calculation of Feynman supergraphs we find that in the $\mathcal{N} = 4$, $\mathcal{N} = 2$, $\mathcal{N} = 1$ and $\mathcal{N} = 0$ SYM theories the one-loop two-particle evolution kernel \mathbb{H}_{12} has the *same*, universal form. Namely, \mathbb{H}_{12} is factorized into a product of two commuting operators

$$\mathbb{H}_{12} = \mathbb{V}_{12} (1 - \Pi_{12}), \quad (5.1)$$

with $[\mathbb{V}_{12}, \Pi_{12}] = 0$. The operator \mathbb{V}_{12} is given by (4.8) for $j = -1/2$ and $\varphi(\xi) = -\delta(\xi)$

$$\begin{aligned} \mathbb{V}_{12}\mathbb{O}(Z_1, \dots, Z_L) = \int_0^1 \frac{d\alpha}{(1-\alpha)\alpha^2} \Big\{ & 2\alpha^2 \mathbb{O}(Z_1, Z_2, \dots, Z_L) \\ & - \mathbb{O}(\alpha Z_1 + (1-\alpha)Z_2, Z_2, \dots, Z_L) - \mathbb{O}(Z_1, \alpha Z_2 + (1-\alpha)Z_1, \dots, Z_L) \Big\}. \end{aligned} \quad (5.2)$$

The second operator is a projector, $(\Pi_{12})^2 = \Pi_{12}$. It is defined as

$$\Pi_{12}\mathbb{O}(Z_1, \dots, Z_L) = \frac{1}{2} (1 + Z_{12} \cdot \partial_Z) \mathbb{O}(Z, Z_2, \dots, Z_L) \Big|_{Z=Z_2} + \frac{1}{2} (1 + Z_{21} \cdot \partial_Z) \mathbb{O}(Z_1, Z, \dots, Z_L) \Big|_{Z=Z_1}, \quad (5.3)$$

where $(Z_{12} \cdot \partial_Z) \equiv (z_1 - z_2)\partial_z + (\theta_1^A - \theta_2^A)\partial_{\theta^A}$. Examining (5.2) we see that the integral over α is divergent for $\alpha \rightarrow 0$. Using (5.2) and (5.3) one can check that divergences cancel in the expression for $[\mathbb{V}_{12}(1 - \Pi_{12}) \cdot \mathbb{O}](Z_1, \dots, Z_L)$ and, therefore, the integral operator (5.1) is well-defined. We would like to stress that Eqs. (5.1) – (5.3) are valid only for the light-cone operators (3.1) built from the chiral superfields in the \mathcal{N} -extended SYM. The \mathcal{N} -dependence enters into Eqs. (5.1) – (5.3) entirely through the dimension of the superspace $Z = (x, \theta^1, \dots, \theta^{\mathcal{N}})$. This means that in order to go, for example, from $\mathcal{N} = 4$ down to $\mathcal{N} = 2$ one has to put $\theta^3 = \theta^4 = 0$.

It is straightforward to verify that the operators \mathbb{V}_{12} and Π_{12} commute with the generators of the superconformal algebra (4.3) and the same is true for \mathbb{H}_{12} . The appearance of the projector $1 - \Pi_{12}$ in the right-hand side of (5.1) can be understood as follows. It can be deduced from the evolution equation (3.6) that, because of this projector, the operator $\Pi_{12}\mathbb{O}(Z_1, Z_2)$ has a vanishing anomalous dimension. Indeed, according to the definition (5.3), the operator $\Pi_{12}\mathbb{O}(Z_1, Z_2)$ contains bilinear products of nonlocal fields, $\partial_+^{-1}A(0)$, $A(0)$ and $\partial_+^{-1}\bar{\lambda}(0)$, which do not contain ultraviolet divergences as long as the Mandelstam-Leibbrandt prescription is used.

The evolution kernel defined in Eqs. (5.1) and (3.7) governs the scale dependence of a nonlocal light-cone operator $\mathbb{O}(Z_1, \dots, Z_L)$, Eq. (3.1). As was already mentioned, $\mathbb{O}(Z_1, \dots, Z_L)$ is a generating function for both the Wilson operators and spurious nonlocal operators. The latter operators can be removed by applying the projector Π , Eq. (3.4), to both sides of the evolution

equation (3.6). One verifies using (5.1) and (5.3) that the operators \mathbb{H}_{12} and Π_{12} satisfy the following relations

$$\Pi\mathbb{H}_{12}(1 - \Pi) = 0, \quad \Pi\Pi_{12} = 0. \quad (5.4)$$

Multiplying both sides of (3.6) by Π one finds that the “physical” light-cone operator $\mathbb{O}^w(Z_1, \dots, Z_L)$, Eq. (3.4), evolves autonomously with the corresponding two-particle evolution kernel given by

$$\mathbb{H}_{12}^w = \Pi\mathbb{H}_{12} = \Pi\mathbb{H}_{12}\Pi = \Pi\mathbb{V}_{12}(1 - \Pi_{12})\Pi = \Pi(1 - \Pi_{12})\mathbb{V}_{12}\Pi = \Pi\mathbb{V}_{12}\Pi. \quad (5.5)$$

Combined together Eqs. (3.7), (5.1), (5.2) and (5.3) define the one-loop evolution kernel for the light-cone operators (3.1) in $\mathcal{N} = 4$, $\mathcal{N} = 2$, $\mathcal{N} = 1$ and $\mathcal{N} = 0$ SYM theories in the multi-color limit. The origin of such universality is the following. According to (4.6), $\varphi(\xi)$ determines the two-particle kernel for maximal helicity operators (4.5). This kernel describes the RG evolution of the light-cone operator $\partial_{x_1}^{-1}A(x_1n_\mu)\partial_{x_2}^{-1}A(x_2n_\mu)$ built from two helicity +1 gauge fields. To one-loop order, the corresponding Feynman diagrams involve cubic and quartic pure gluonic vertices and, therefore, they are only sensitive to the pure gluonic part of the \mathcal{N} -extended SYM action. The latter is the same for the light-cone actions (2.1), (2.8), (2.10) and (2.12).

Expanding both sides of (3.6) in powers of even and odd variables, z_k and θ_k^A , respectively, one can obtain from (5.1) the mixing matrix for Wilson operators of the maximal Lorentz spin in the \mathcal{N} -extended SYM. To illustrate the predictive power of (5.1) let us derive the mixing matrices for three different sets of the Wilson operators: maximal helicity gauge field operators, scalar operators in the $\mathcal{N} = 4$ SYM theory with no derivatives and twist-two $SO(6)$ singlet operators involving an arbitrary number of derivatives. Obviously, these three examples do not cover all possible Wilson operators. A general classification of the solutions to (3.6) will be given elsewhere.

(i) Maximal helicity Wilson gauge operators are defined in the light-cone gauge $A_+(x) = 0$ as

$$O_{j_1 j_2 \dots j_L}^h(0) = \text{tr}\{\partial_+^{j_1+1}A(0)\partial_+^{j_2+1}A(0)\dots\partial_+^{j_L+1}A(0)\}, \quad (5.6)$$

with $0 \leq j_k < \infty$ counting the number of derivatives. To one-loop order, in the multi-color limit, the operators (5.6) mix under renormalization with the operators $O_{j'_1 j'_2 \dots j'_L}^h(0)$ having the same total number of derivatives $j'_1 + \dots + j'_L$. According to (2.2), the helicity +1 gauge field determines the lowest component of the $\mathcal{N} = 4$ superfield leading to $\partial_+^{j+1}A(x) = \partial_x^{j+2}\Phi(x, 0, 0)$. Therefore, applying $\partial_{z_1}^{j_1+2}\dots\partial_{z_L}^{j_L+2}$ to both sides of (3.6) and putting $Z_1 = \dots = Z_L = 0$ afterwards, one obtains from (3.6) the evolution equation for the Wilson operators (5.6). The corresponding mixing matrix takes the form (see Eq. (3.7))

$$[\mathbb{H}]_{j_1 \dots j_L}^{j'_1 \dots j'_L} = V_{j_1 j_2}^{j'_1 j'_2} \delta_{j_3}^{j'_3} \dots \delta_{j_L}^{j'_L} + \dots + \delta_{j_1}^{j'_1} \delta_{j_2}^{j'_2} \dots V_{j_{L-1} j_L}^{j'_{L-1} j'_L} + \delta_{j_2}^{j'_2} \delta_{j_3}^{j'_3} \dots V_{j_L j_1}^{j'_L j'_1}, \quad (5.7)$$

where the two-particle mixing matrix $V_{j_1 j_2}^{j'_1 j'_2}$ describes the transition $\partial_+^{j_1+1}A\partial_+^{j_2+1}A \rightarrow \partial_+^{j'_1+1}A\partial_+^{j'_2+1}A$. It is related to the two-particle evolution kernel as

$$\partial_{z_1}^{j_1+2}\partial_{z_2}^{j_2+2}[\mathbb{H}_{12} \cdot \mathbb{O}](Z_1, Z_2) \Big|_{Z_1=Z_2=0} = \sum_{j'_1+j'_2=j_1+j_2} V_{j_1 j_2}^{j'_1 j'_2} \partial_+^{j'_1+1}A(0)\partial_+^{j'_2+1}A(0). \quad (5.8)$$

This relation establishes the correspondence between the mixing matrix of local Wilson operators and evolution kernels of nonlocal light-cone operators. Making use of (5.1), the left-hand side of

(5.8) can be written after some algebra as

$$\text{Eq. (5.8)} = \int_0^1 \frac{d\alpha}{\bar{\alpha}} \alpha^2 \left[2\partial_{z_1}^{j_1} \partial_{z_2}^{j_2} - \alpha^{j_1} (\bar{\alpha} \partial_{z_1} + \partial_{z_2})^{j_2} \partial_{z_1}^{j_1} - \alpha^{j_2} (\bar{\alpha} \partial_{z_2} + \partial_{z_1})^{j_1} \partial_{z_2}^{j_2} \right] \partial_{z_1} A(z_1) \partial_{z_2} A(z_2) \Big|_{z_k=0}. \quad (5.9)$$

This expression serves as a generating function for the matrix elements $V_{j_1 j_2}^{j'_1 j'_2}$ entering (5.8). The mixing matrix (5.7) for the maximal helicity operators (5.6) with an arbitrary number of derivatives and a given number of fields L has an infinite size. In addition, as was mentioned in Sect. 1, it possesses a hidden symmetry. Namely, the evolution kernel (5.7) can be mapped into a Hamiltonian of completely integrable Heisenberg $SL(2)$ spin chain of length L .

(ii) Composite scalar operators are built from six real scalar fields $\phi_j(x)$ ($j = 1, \dots, 6$)

$$O_{j_1 j_2 \dots j_L} = \text{tr} \{ \phi_{j_1}(0) \phi_{j_2}(0) \dots \phi_{j_L}(0) \}, \quad \phi_j(x) = \frac{1}{2\sqrt{2}} \Sigma_j^{AB} \bar{\phi}_{AB}(x), \quad (5.10)$$

which are given by linear combinations of $\bar{\phi}_{AB}$. Here, $j = 1, \dots, 6$ and Σ_j^{AB} are the chiral blocks of Dirac matrices in six-dimensional Euclidean space [29]. In the multi-color limit, to one-loop order the scalar operators $O_{j_1 \dots j_L}(0)$ mix with each other under renormalization. Similar to the previous case, one uses the relation between the scalar fields and the $\mathcal{N} = 4$ superfield (2.2), $\phi_j(x) = \frac{i}{2\sqrt{2}} \Sigma_j^{AB} \partial_{\theta^A} \partial_{\theta^B} \Phi(x, \theta^A, 0)|_{\theta^A=0}$, to derive from (3.6) the evolution equation for the scalar operators (5.10). The corresponding mixing matrix takes the same form as before, Eq. (5.7), but the expression for the two-particle mixing matrix $V_{j_1 j_2}^{j'_1 j'_2}$ describing the transition $\phi_{j_1} \phi_{j_2} \rightarrow \phi_{j'_1} \phi_{j'_2}$ is different

$$-\frac{1}{8} \left(\Sigma_{j_1}^{AB} \partial_{\theta_1^A} \partial_{\theta_1^B} \right) \left(\Sigma_{j_2}^{CD} \partial_{\theta_2^C} \partial_{\theta_2^D} \right) [\mathbb{H}_{12} \cdot \mathbb{O}](Z_1, Z_2) \Big|_{Z_1=Z_2=0} = \sum_{j'_1 j'_2} V_{j_1 j_2}^{j'_1 j'_2} \phi_{j'_1}(0) \phi_{j'_2}(0). \quad (5.11)$$

Substituting (5.1) into this relation one finds after some algebra

$$V_{j_1 j_2}^{j'_1 j'_2} = \delta_{j_1}^{j'_1} \delta_{j_2}^{j'_2} + \frac{1}{2} \delta_{j_1 j_2} \delta^{j'_1 j'_2} - \delta_{j_1}^{j'_2} \delta_{j_2}^{j'_1} = \left| \right| + \frac{1}{2} \bigcup - \bigtimes. \quad (5.12)$$

The expressions (5.7) and (5.12) define the one-loop evolution kernel for the scalar operators (5.10) in the multi-color limit. They are in agreement with the results of Ref. [10]. The mixing matrix $[\mathbb{H}]_{j_1 \dots j_L}^{j'_1 \dots j'_L}$ for the product of L scalar operators has a finite dimension 6^L . As in the previous case, it has a hidden symmetry – this matrix can be mapped into a Hamiltonian of completely integrable Heisenberg $SO(6)$ spin chain of length L [10].

(iii) Twist-two $SO(6)$ singlet parity-even gauge operators are defined as

$$O^g(N) = \text{tr} \{ F_{+\mu} D_+^N F_+{}^\mu \} = \text{tr} \{ \partial_+^{N+1} \bar{A} \partial_+ A + \partial_+ \bar{A} \partial_+^{N+1} A \}, \quad (5.13)$$

where in the second relation we adopted the light-cone gauge $A_+(x) = 0$. They mix under renormalization with the twist-two fermion and scalar operators of the same canonical dimension

$$O^q(N) = \text{tr} \{ \partial_+^{N+1} \bar{\lambda}_A \lambda^A - \bar{\lambda}_A \partial_+^{N+1} \lambda^A \}, \quad O^s(N) = \text{tr} \{ \bar{\phi}_{AB} \partial_+^{N+2} \phi^{AB} \}, \quad (5.14)$$

as well as with the operators containing total derivatives $\partial_+^n O^a(N-n)$, with $a = g, q, s$ and $n = 1, \dots, N$. The latter operators can be effectively eliminated by going over to the forward matrix

elements of the operators (5.13) and (5.14). The mixing between the operators (5.13) and (5.14) is described by a 3×3 matrix of the anomalous dimensions $\gamma_{ab}(N)$ (with $a, b = g, q, s$). To calculate its one-loop expression from (3.6), one uses the relations $\partial_+ A(x) = \partial_x^2 \Phi(x, 0, 0)$ and $\partial_+ \bar{A}(x) = -d_\theta \Phi(x, 0, 0)$, where the $\mathcal{N} = 4$ superfield is given by (2.2) and $d_\theta \equiv \frac{1}{4!} \varepsilon^{ABCD} \partial_{\theta^A} \partial_{\theta^B} \partial_{\theta^C} \partial_{\theta^D}$. Then, the evolution equation for the gauge field operator (5.13) can be derived from (3.6) as

$$\mu \frac{d}{d\mu} \langle O^g(N) \rangle = \frac{g^2 N_c}{8\pi^2} \langle (\partial_{z_1}^N + \partial_{z_2}^N) d_{\theta_1} \partial_{z_2}^2 [2\mathbb{H}_{12} \cdot \mathbb{O}] (Z_1, Z_2) \rangle \Big|_{Z_k=0} = \frac{g^2 N_c}{8\pi^2} \sum_{b=g,q,s} \gamma_{gb}(N) \langle O^b(N) \rangle, \quad (5.15)$$

where $\langle \dots \rangle$ stands for the forward matrix element and the additional factor of 2 takes into account that the evolution kernel for the two-particle operators equals $\mathbb{H} = \mathbb{H}_{12} + \mathbb{H}_{21} = 2\mathbb{H}_{12}$. Substituting (5.1) into (5.15) one finds after some algebra

$$\begin{aligned} \gamma_{gg}(N) &= 4 \left[\psi(1) - \psi(N+1) - \frac{2}{N+2} + \frac{1}{N+3} - \frac{1}{N+4} \right], \\ \gamma_{gq}(N) &= \frac{4}{N+1} - \frac{4}{N+2} + \frac{2}{N+3}, \quad \gamma_{gs}(N) = \frac{2}{(N+1)(N+2)}, \end{aligned} \quad (5.16)$$

where $\psi(x) = d \ln \Gamma(x) / dx$ is the Euler function. The remaining entries of the matrix of anomalous dimensions can be calculated in a similar manner. This leads to the expression for $\gamma_{ab}(N)$ which is in agreement with the results of Ref. [30]. As we will show in the next section, $\gamma_{ab}(N)$ can be mapped into a two-particle Hamiltonian of the Heisenberg $SL(2|4)$ spin chain.

6. Heisenberg $SL(2|\mathcal{N})$ spin chain

Let us show that the one-loop evolution kernel for light-cone operators (3.1) in the \mathcal{N} -extended SYM theory defined in (3.7) and (5.1) possesses a hidden integrability – it can be identified as a Hamiltonian of the Heisenberg $SL(2|\mathcal{N})$ spin chain. To this end, we shall construct the R -matrix on the space of light-cone superfields and argue that its logarithmic derivative coincides with the expression for the two-particle evolution kernel (5.5).

The light-cone scalar superfield $\Phi(z, \theta^A)$ defines a representation of the superconformal $SL(2|\mathcal{N})$ algebra that we shall denote as V . The fact that $\Phi(z, \theta^A)$ is chiral implies that V is the so-called atypical representation [28]. The generators of the algebra are realized on V as differential operators (4.3). The states $\{1, z, \theta^A, z\theta^A, \theta^A\theta^B, \dots\} \in V$ define coefficient functions in the expansion of the superfield around the origin in the superspace $\Phi(z, \theta^A) = \Phi(0, 0) + Z \cdot \partial_Z \Phi(0, 0) + \frac{1}{2!} (Z \cdot \partial_Z)^2 \Phi(0, 0) + \dots$ with 1 being the lowest weight in V . The quadratic Casimir operator is [28]⁵

$$\mathbb{J}^2 = (L^0)^2 + L^+ L^- + (\mathcal{N} - 1) L^0 + \frac{\mathcal{N}}{\mathcal{N} - 2} B^2 - V_A^+ W^{A,-} - W_A^+ V^{A,-} - \frac{1}{2} T^B{}_A T^A{}_B. \quad (6.1)$$

For the superfield $\Phi(z, \theta^A)$ one has $\mathbb{J}^2 \Phi(z, \theta^A) = j \left[j + \mathcal{N} - 1 + \frac{\mathcal{N}j}{\mathcal{N} - 2} \right] \Phi(z, \theta^A)$, with $j = -1/2$ being the conformal spin of the superfield (4.4).

A distinguished feature of V is that for $j = -1/2$ it contains an invariant subspace spanned by the vectors $V_0 = \{1, z, \theta^1, \dots, \theta^\mathcal{N}\}$. The corresponding superfield $\Phi_0(z, \theta^A) = \Phi(0, 0) +$

⁵The singularity of \mathbb{J}^2 at $\mathcal{N} = 2$ is spurious since it can be removed by adding to the r.h.s. of (6.1) an infinite c-number correction.

$z\partial_z\Phi(0,0) + \theta^A\partial_{\theta^A}\Phi(0,0)$ is built entirely from spurious components of fields and is annihilated by the projector (3.4), $\Pi\Phi_0(z, \theta^A) = 0$. Thus, the “physical” superfield $\Phi^w = \Phi(z, \theta^A) - \Phi_0(z, \theta^A)$ defined in Eq. (3.5) belongs to the quotient of the two spaces $V_w = V/V_0$. Let us now consider the product of two superfields $\Phi(Z_1)\Phi(Z_2)$ belonging to the tensor product $V \otimes V$. The generators of the superconformal algebra are given on $V \otimes V$ by the sum of the differential operators (4.3) acting on the Z_1 - and Z_2 -coordinates. Using (6.1) one can define the corresponding two-particle Casimir operator \mathbb{J}_{12}^2 and realize it on $V \otimes V$ as a 2×2 matrix $[\mathbb{J}_{12}^2]_{11} = \Pi\mathbb{J}_{12}^2\Pi$, $[\mathbb{J}_{12}^2]_{12} = \Pi\mathbb{J}_{12}^2(1 - \Pi)$, etc. Since $(1 - \Pi)V \otimes V = V_0 \otimes V_0 - V_0 \otimes V - V \otimes V_0$ and V_0 is the invariant subspace annihilated by the projector Π , one has $[\mathbb{J}_{12}^2]_{12} = 0$, or equivalently,

$$\Pi\mathbb{J}_{12}^2(1 - \Pi)\Phi(Z_1)\Phi(Z_2) = 0. \quad (6.2)$$

This relation implies that the Casimir operator is given a triangular matrix $\mathbb{J}_{12}^2 = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. The same is true for an arbitrary two-particle operator like \mathbb{H}_{12} (see Eq. (5.4)) invariant under the superconformal transformations.

To reveal integrable structures of the evolution equations (3.6) in the \mathcal{N} -extended SYM theory, we apply the R -matrix approach [31]. As the starting point, we introduce the Lax operator for the $SL(2|\mathcal{N})$ algebra [32, 33, 34]. It is given by the graded matrix of dimension $\mathcal{N} + 2$ whose entries are linear combinations of the generators of the superconformal algebra (4.3)

$$[\mathbb{L}(u)]_{ab} = \begin{pmatrix} u + L^0 + \frac{\mathcal{N}}{\mathcal{N}-2}B & -W^{B,-} & L^- \\ -V_A^+ & u\delta_A^B - T_A^B + \frac{2}{\mathcal{N}-2}B\delta_A^B & V_A^- \\ L^+ & -W^{B,+} & u - L^0 + \frac{\mathcal{N}}{\mathcal{N}-2}B \end{pmatrix}, \quad (6.3)$$

where u is a complex spectral parameter and the indices run over $a = (0, A, \mathcal{N} + 1)$ and $b = (0, B, \mathcal{N} + 1)$ with $A, B = 1, \dots, \mathcal{N}$.

Let us now define an integral operator $\mathbb{R}_{12}(u)$ acting on the tensor product $V \otimes V$ as

$$\begin{aligned} [\mathbb{R}_{12}(u) \cdot \mathbb{O}](Z_1, Z_2) &= \frac{u \sin(\pi u)}{\pi} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta (\alpha\beta)^{-u-1} (1 - \alpha - \beta)^{2j+u-1} \\ &\quad \times \mathbb{O}(\alpha Z_1 + (1 - \alpha)Z_2, \beta Z_2 + (1 - \beta)Z_1), \end{aligned} \quad (6.4)$$

where $j = -1/2$ and u is a spectral parameter. One can verify that the operator (6.4) satisfies the $SL(2|\mathcal{N})$ Yang-Baxter equation

$$\mathbb{R}_{12}(u)\mathbb{R}_{13}(u+v)\mathbb{R}_{23}(v) = \mathbb{R}_{23}(v)\mathbb{R}_{13}(u+v)\mathbb{R}_{12}(u) \quad (6.5)$$

and interchanges the Lax operators $\mathbb{L}_1(u) = \mathbb{L}(u) \otimes \mathbb{1}$ and $\mathbb{L}_2(u) = \mathbb{1} \otimes \mathbb{L}(u)$

$$\mathbb{L}_1(u)\mathbb{L}_2(u+v)\mathbb{R}_{12}(v) = \mathbb{R}_{12}(v)\mathbb{L}_2(u+v)\mathbb{L}_1(u). \quad (6.6)$$

Here, the Lax operators $\mathbb{L}_k(u)$ (with $k = 1, 2$) are given by (6.3) with the superconformal charges acting on the Z_k -coordinates. If one puts in (6.4) the odd coordinates equal to zero, $Z_k = (z_k, \theta_k^A = 0)$, the operator $\mathbb{R}_{12}(u)$ is reduced to the known expression for the $SL(2)$ invariant R -matrix acting along the z -axis in the superspace [34].

The relation (6.4) can be seen as a special case of the $SL(2|\mathcal{N})$ invariant operator (4.8) for $\varphi(\xi) = (\xi - 1)^{2j+u-1}u \sin(\pi u)/\pi$. The operator $\mathbb{R}_{12}(u)$ satisfies the same relation (6.2) as the two-particle Casimir operator \mathbb{J}_{12}^2 and, therefore, it is given by a triangular matrix

$$\mathbb{R}_{12}(u) = \begin{pmatrix} \mathbb{R}_{12}^w(u) & 0 \\ * & * \end{pmatrix}, \quad \mathbb{R}_{12}^w(u) \equiv \Pi \mathbb{R}_{12}(u) = \Pi \mathbb{R}_{12}(u) \Pi. \quad (6.7)$$

Substituting this expression into (6.5) one finds that $\mathbb{R}_{12}^w(u)$ satisfies the Yang-Baxter equation (6.5). Thus, the operator $\mathbb{R}_{12}^w(u)$ is the R -matrix on the tensor product of “physical” spaces $V_w \otimes V_w$ with $V_w \equiv V \setminus V_0$. Invoking the standard arguments [31], one finds that its logarithmic derivative at $u = 0$ defines the two-particle Hamiltonian of a completely integrable, homogenous XXX Heisenberg $SL(2|\mathcal{N})$ spin chain

$$\mathbb{H}_{12}^{\text{xxx}} = \frac{d}{du} \ln \mathbb{R}_{12}^w(u) \Big|_{u=0} = \Pi \left(\frac{d}{du} \ln \mathbb{R}_{12}(u) \Big|_{u=0} \right) \Pi. \quad (6.8)$$

To calculate the derivative entering this expression, one expands the integral in the right-hand side of (6.4) at small u and projects the result onto Π , Eq. (3.4). In this way, one finds after some algebra

$$\mathbb{R}_{12}^w(u) = \mathbb{P}_{12} [1 + u \cdot \Pi \mathbb{V}_{12} \Pi + \mathcal{O}(u^2)] , \quad (6.9)$$

where the integral operator \mathbb{V}_{12} was defined in (5.2) and \mathbb{P}_{12} is the permutation operator, $\mathbb{P}_{12}\mathbb{O}(Z_1, Z_2) = \mathbb{O}(Z_2, Z_1)$. Substituting this relation into (6.8) one recovers the two-particle evolution kernel (5.5), $\mathbb{H}_{12}^{\text{xxx}} = \mathbb{H}_{12}^w$. This allows us to identify the evolution kernel for the “physical” light-cone operators $\mathbb{O}^w(Z_1, \dots, Z_L)$ as the Hamiltonian of a *noncompact* Heisenberg $SL(2|\mathcal{N})$ spin chain of length L and the single-particle spin $j = -1/2$. Similar integrable spin chains have been previously studied in Refs. [34].

To understand the properties of the obtained expression for the R -matrix one chooses $j = -\frac{1}{2} + \epsilon$ and examines Eqs. (6.4) and (6.8) in the limit $\epsilon \rightarrow 0$. For $\epsilon \neq 0$, the $SL(2|\mathcal{N})$ generators are given by (4.3). The corresponding representation space $V(\epsilon)$ is irreducible for $\epsilon \neq 0$ while at $\epsilon = 0$ one has $V(0) = V_0 \oplus V_w$. The tensor product $V(\epsilon) \otimes V(\epsilon)$ can be decomposed into irreducible components with the lowest weights $\Psi_l(Z_1, Z_2)$ ($l = 0, 1, \dots$)

$$\Psi_0 = 1, \quad \Psi_k = \theta_{12}^{A_1} \dots \theta_{12}^{A_k}, \quad \Psi_{n+\mathcal{N}} = \varepsilon_{A_1 \dots A_{\mathcal{N}}} \theta_{12}^{A_1} \dots \theta_{12}^{A_{\mathcal{N}}} z_{12}^n, \quad (6.10)$$

where $\theta_{12}^A = \theta_1^A - \theta_2^A$, $z_{12} = z_1 - z_2$, $1 \leq k \leq \mathcal{N} - 1$ and $0 \leq n < \infty$. These states satisfy the relations $L^- \Psi_l = W^{A,-} \Psi_l = V_A^- \Psi_l = 0$ and diagonalize the two-particle Casimir operator (6.1)

$$(\mathbb{J}_{12}^2 - \Delta_j) \Psi_l = (l + 2j)(l + 2j - 1) \Psi_l = J_{12}(J_{12} - 1) \Psi_l, \quad (6.11)$$

with $\Delta_j = 2j\mathcal{N} [1 + \frac{2j}{\mathcal{N}-2}]$. Here the notation was introduced for the two-particle superconformal spin $J_{12} = l + 2j = -1 + l + 2\epsilon$ with $l = 0, 1, \dots$. Substituting the lowest weights into (6.4), one evaluates the eigenvalues of the R -matrix

$$\mathbb{R}_{12}(u) \Psi_l = (-1)^l \frac{\Gamma(1-u)\Gamma(J_{12}+u)}{\Gamma(1+u)\Gamma(J_{12}-u)} \Psi_l. \quad (6.12)$$

For the modules with the lowest weights Ψ_0 and Ψ_1 the eigenvalues of $[d \ln \mathbb{R}_{12}(u)/du]_{u=0}$ behave as $\sim 1/\epsilon$ while for the remaining modules Ψ_l ($l \geq 2$) it approaches a finite value as $\epsilon \rightarrow 0$. Notice

that the projector Π annihilates the modules Ψ_0 and Ψ_1 and, therefore, the Hamiltonian (6.8) is well-defined for $\epsilon \rightarrow 0$

$$\mathbb{H}_{12}^w \Psi_{l+2}^w = 2 [\psi(l+1) - \psi(1)] \Psi_{l+2}^w, \quad (6.13)$$

where $\Psi_l^w = \Pi \Psi_l$ and $\Psi_0^w = \Psi_1^w = 0$. Equation (6.13) defines the eigenvalues of the two-particle evolution kernel (5.5) as a function of the superconformal spin $J_{12} = l - 1$.

The Hamiltonian of the Heisenberg $SL(2|\mathcal{N})$ spin magnet possesses a set of the integrals of motion. In the R -matrix approach, one can find their explicit form by constructing the auxiliary transfer matrix. It is equal to the supertrace of the product of the Lax operators (6.3) and is given by a polynomial in u of degree L with operator-valued coefficients,

$$t_L(u) = \text{str} \{ \mathbb{L}_L(u) \dots \mathbb{L}_2(u) \mathbb{L}_1(u) \} = (2 - \mathcal{N})u^L + q_2 u^{L-2} + \dots + q_L, \quad (6.14)$$

where $q_2 = \mathbb{J}_{12\dots L}^2 - L[\frac{1}{2}\Delta_{-1/2} + \frac{3}{4}]$ is related to the total superconformal spin. It follows from the Yang-Baxter equations (6.5) and (6.6) that the operators q_k commute among themselves and with the evolution kernel \mathbb{H} . The spectral problem for the Heisenberg $SL(2|\mathcal{N})$ spin magnet can be solved within the Bethe Ansatz (see, e.g., [35]). Using its eigenspectrum one can construct the basis of superconformal operators having an autonomous scale dependence and evaluate the corresponding anomalous dimensions. We shall return to this problem in a forthcoming publication.

7. Conclusions

In this paper, we have demonstrated that the one-loop evolution kernel governing the scale dependence of the single trace product of chiral light-cone superfields in the \mathcal{N} -extended SYM theory coincides in the multi-color limit with the Hamiltonian of the Heisenberg $SL(2|\mathcal{N})$ spin chain. We constructed the evolution kernel as an integral operator acting on the superspace and found that it has the same, universal form in the $\mathcal{N} = 0$, $\mathcal{N} = 1$, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM theories. The only difference is that going over from the $\mathcal{N} = 0$ to the maximally symmetric $\mathcal{N} = 4$ theory, one has to increase the number of odd dimensions in the superspace.

We already mentioned that in multi-color QCD the one-loop dilatation operator acting on the Wilson operators of maximal helicity built from the field strength tensor and sandwiched between the quark fields in the fundamental representation coincides with the Hamiltonian of the open Heisenberg $SL(2)$ spin chain. It is straightforward to lift this structure into the superspace and define the corresponding $SL(2|\mathcal{N})$ invariant dilatation operator. It is natural to expect that it describes the scale dependence of the product of the light-cone superfields in the \mathcal{N} -extended SYM theory supplemented with the matter in the fundamental representation.

In our analysis we restricted ourselves to Wilson operators of the maximal Lorentz spin built from the “good” components of the fields. We recall that in the light-cone formalism, supersymmetry is realized differently for the “good” and “bad” components. In our consideration we made use of a part of superconformal generators which admit a linear realization on the superspace. Making use of the remaining part of superconformal generators, one can extend the analysis to the Wilson operators of lower spin and those built from “bad” components.

The dilatation operator constructed in this paper acts on the space of a single trace product of chiral superfields. In the $\mathcal{N} = 4$ SYM it covers all possible Wilson operators of the maximal Lorentz spin while in the SYM theory with less supersymmetry it should be supplemented by mixed products of chiral and antichiral light-cone superfields. In the latter case, the dilatation

operator can be realized as a Hamiltonian of the spin chain but its integrability property will be lost. As was shown in Refs. [4, 5], the additional terms in the Hamiltonian responsible for breaking the integrability lead to the formation of a *mass gap* in the spectrum of the anomalous dimensions in multi-color QCD. This issue deserves further investigation in the \mathcal{N} -extended SYM theory.

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References

- [1] L.N. Lipatov, JETP Lett. 59 (1994) 596.
- [2] L.D. Faddeev, G.P. Korchemsky, Phys. Lett. B 342 (1995) 311.
- [3] V.M. Braun, S.É. Derkachov, A.N. Manashov, Phys. Rev. Lett. 81 (1998) 2020.
- [4] V.M. Braun, S.É. Derkachov, G.P. Korchemsky, A.N. Manashov, Nucl. Phys. B 553 (1999) 355.
- [5] A.V. Belitsky, Phys. Lett. B 453 (1999) 59; Nucl. Phys. B 558 (1999) 259; Nucl. Phys. B 574 (2000) 407.
- [6] S.É. Derkachov, G.P. Korchemsky, A.N. Manashov, Nucl. Phys. B 566 (2000) 203.
- [7] A.S. Gorsky, I.I. Kogan, G. Korchemsky, J. High Ener. Phys. 0205 (2002) 053;
A.V. Belitsky, A.S. Gorsky, G.P. Korchemsky, Nucl. Phys. B 667 (2003) 3.
- [8] Yu.M. Makeenko, Sov. J. Nucl. Phys. 33 (1981) 440;
T. Ohrndorf, Nucl. Phys. B 198 (1982) 26.
- [9] A.P. Bukhvostov, G.V. Frolov, L.N. Lipatov, E.A. Kuraev, Nucl. Phys. B 258 (1985) 601.
- [10] J.A. Minahan, K. Zarembo, J. High Ener. Phys. 03 (2003) 013.
- [11] N. Beisert, C. Kristjansen, M. Staudacher, Nucl. Phys. B 664 (2003) 131.
- [12] N. Beisert, Nucl. Phys. B 676 (2004) 3;
N. Beisert, M. Staudacher, Nucl. Phys. B 670 (2003) 439.
- [13] B. Stefanski, hep-th/0312091.
- [14] X.J. Wang, Y.S. Wu, hep-th/0311073.
- [15] B. Chen, X.J. Wang, Y.S. Wu, hep-th/0401016;
O. DeWolfe, N. Mann, hep-th/0401041.
- [16] J.B. Kogut, D.E. Soper, Phys. Rev. D 1 (1970) 2901.

- [17] L. Brink, O. Lindgren, B.E. Nilsson, Nucl. Phys. B 212 (1983) 401; Phys Lett. B 123 (1983) 323.
- [18] S. Mandelstam, Nucl. Phys. B 213 (1983) 149; Phys. Lett. B 121 (1983) 30.
- [19] M.F. Sohnius, P.C. West, Phys. Lett. B 100 (1981) 245.
- [20] P.S. Howe, K.S. Stelle, P.K. Townsend, Nucl. Phys. B 236 (1984) 125.
- [21] L. Brink, A. Tollstén, Nucl. Phys. B 249 (1985) 244.
- [22] A. Smith, Nucl. Phys. B 261 (1985) 285.
- [23] S.É. Derkachov, R. Kirschner, Phys. Rev. D 64 (2001) 074013.
- [24] I.I. Balitsky, V.M. Braun, Nucl. Phys. B 311 (1989) 541.
- [25] V.M. Braun, G.P. Korchemsky, D. Müller, Prog. Part. Nucl. Phys. 51 (2003) 311.
- [26] D. Müller, Phys. Rev. D 49 (1994) 2525;
A.V. Belitsky, D. Müller, Nucl. Phys. B 537 (1999) 397.
- [27] S.J. Gates, M.T. Grisaru, M. Rocek, W. Siegel, Front. Phys. 58 (1983) 1;
M.F. Sohnius, Phys. Rept. 128 (1985) 39.
- [28] M. Scheunert, W. Nahm, V. Rittenberg, J. Math. Phys. 18 (1977) 146;
P.D. Jarvis, H.S. Green, J. Math. Phys. 20 (1979) 2115;
L. Frappat, P. Sorba, A. Sciarrino, hep-th/9607161.
- [29] A.V. Belitsky, S.É. Derkachov, G.P. Korchemsky, A.N. Manashov, hep-th/0311104.
- [30] A.V. Kotikov, L.N. Lipatov, Nucl. Phys. B 661 (2003) 19.
- [31] L.A. Takhtajan, L.D. Faddeev, Russ. Math. Survey 34 (1979) 11;
E.K. Sklyanin, L.A. Takhtajan, L.D. Faddeev, Theor. Math. Phys. 40 (1980) 688;
V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge Univ. Press, 1993.
- [32] P.P. Kulish, J. Sov. Math. 35 (1986) 2648; J. Sov. Math. 35 (1986) 1111; hep-th/9507070.
- [33] H. Frahm, M.P. Pfannmüller, A.M. Tselik, Phys. Rev. Lett. 81 (1998) 2116;
H. Frahm, Nucl. Phys. B 559 (1999) 613.
- [34] S.É. Derkachov, D. Karakhanyan, R. Kirschner, Nucl. Phys. B 583 (2000) 691; Nucl. Phys. B 618 (2001) 589.
- [35] H. Saleur, Nucl. Phys. B 578 (2000) 552.